Holographic Stress Tensors for Kerr–AdS Black Holes

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(September 1999)

We use the counterterm subtraction method to calculate the action, stress–energy–momentum tensor for the (Kerr) rotating black holes in AdS$_{n+1}$, for n=2,3 and 4. We demonstrate that the expressions for the total energy for the Kerr–AdS$_3$ and Kerr–AdS$_5$ spacetimes, in the limit of vanishing black hole mass, are equal to the Casimir energies of the holographically dual n–dimensional conformal field theories. In particular, for Kerr–AdS$_5$, dual to the case of four dimensional N=4 supersymmetric Yang–Mills theory on the rotating Einstein universe, we explicitly verify the equality of the T=0 stress tensor from the two sides of the correspondence, and present the result for general T as a prediction from gravity. Amusingly, in this case it is observed that while the trace of the stress tensor does not vanish, its integral does, thereby keeping the action free of ultraviolet divergences.

I. INTRODUCTION

The AdS/CFT correspondence relates an $(n+1)$–dimensional theory of gravity on anti–de Sitter (AdS) spacetime (times a compact manifold) to a conformal field theory (CFT) in n dimensions. This duality first arose as a result of investigating [1] N parallel D3–branes in the context of the low energy, (*i.e.*, the limit of zero α', the inverse string tension) classical (weak string coupling, g_s) limit of type IIB superstring theory on AdS$_5\times S^5$. A precise statement of the AdS/CFT correspondence [2,3] is the equality of the partition functions of the two theories,

$$Z_{AdS}(\phi_i) = Z_{CFT}(\phi_{0,i}) \ . \tag{1}$$

From the gravity–on–AdS point of view, ϕ_i is a bulk field constrained to the values $\phi_{0,i}$ on the boundary, while from the CFT point of view, $\phi_{0,i}$ are sources for pointlike operators, $\mathcal{O}_i$, in the theory. In the low energy limit of the theory one can use the classical gravitational action to calculate the partition function of the CFT on the boundary. This action has the form [4],

$$\begin{aligned} I_{\rm bulk} + I_{\rm surf} = & -\frac{1}{16\pi G}\int_{\mathcal{M}} d^{n+1}x\sqrt{-g}\left(R + \frac{n(n-1)}{l^2}\right) \\ & -\frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^n x\sqrt{-h}\, K. \end{aligned} \tag{2}$$

The first term is the Einstein–Hilbert action with negative cosmological constant ($\Lambda{=}-n(n{-}1)/2l^2$). The second term is the Gibbons–Hawking boundary term. Here, h_{ab} is the boundary metric and K is the trace of the extrinsic curvature K^{ab} of the boundary.

To deal with the divergences which appear in the gravitational action (arising from integrating over the infinite volume of spacetime), two different techniques may be employed. The "traditional" background subtraction technique [4,5] (which subtracts the contribution from a reference spacetime to get a finite result) and the "counterterm subtraction" method [6] (which regulates the action by the addition of certain boundary counterterms). As the counterterms depend upon the geometrical properties of the boundary of the spacetime, the counterterm subtraction method provides an intrinsic definition of the action for a particular spacetime. This sidesteps problems which arise in using the other method when the spacetime in question has an ambiguous (or simply unknown) choice of background.

The divergences of the Einstein–Hilbert action (in dimensions less than six) can be canceled by adding the following counterterms[1],

$$I_{\rm ct} = \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^n x\sqrt{-h}\left[\frac{(n-1)}{l} - \frac{l\mathcal{R}}{2(n-1)}\right]. \tag{3}$$

Here $\mathcal{R}$ is the Ricci scalar for the boundary metric h. Using these counterterms one can construct a divergence free stress tensor which is given by ($I{=}I_{\rm bulk}{+}I_{\rm surf}{+}I_{\rm ct}$):

$$\begin{aligned} T^{ab} = & \frac{2}{\sqrt{-h}}\frac{\delta I}{\delta h_{ab}} = \\ & \frac{1}{8\pi G}\left[K^{ab} - h^{ab}K - \frac{(n-1)}{l}h^{ab} + \frac{lG^{ab}}{(n-2)}\right] \ , \end{aligned} \tag{4}$$

[1] In ref. [7], the counterterm subtraction method was studied extensively, with many examples, and a new counterterm was presented there which allows the subtraction regularization to be performed in dimensions $n{+}1{=}6,7$. See also refs. [8–14] for related studies, some more applications and further extensions.

where G^{ab} is the Einstein tensor in n dimensions. (Correspondingly, the last term should be omitted for n=2.)

The above prescription gives a definition of the action and stress–tensor on any region (say, of radius r) bounding the interior of AdS. The AdS/CFT holographic relation (in the form which we will need it here) equates these quantities to a dual conformal field theory residing on the boundary at infinity[2] ($r\to\infty$ in the coordinates we will choose later). Recalling that the metric restricted to the boundary, h_{ab}, diverges due to an infinite conformal factor r^2/l^2, we define the background metric upon which the dual field theory resides as

$$\gamma_{ab} = \lim_{r\to\infty} \frac{l^2}{r^2} h_{ab} \ . \tag{5}$$

Consequently, the field theory's stress–tensor, $\widehat{T}^{ab}$, is related to the one above (4) by the rescaling [15]:

$$\sqrt{-\gamma}\,\gamma_{ab}\widehat{T}^{bc} = \lim_{r\to\infty} \sqrt{-h}\,h_{ab}T^{bc} \ , \tag{6}$$

which amounts to multiplying all expressions for T^{ab} displayed later by $(r/l)^{n-2}$ before taking the limit $r\to\infty$.

II. KERR–ADS$_3$

As a warm–up and review, we will study the case of Kerr–AdS$_3$. The explicit computation of the energy–momentum tensor and the mass, angular momentum and Casimir energy using the counterterm subtraction technique has been done in ref. [6]. Here, we will perform all of those computations in a different choice of coordinates, supplementing the discussion and calculations where necessary. This will serve the twin purposes of setting up the notation of the rest of the paper, and illustrating the similarities to (and differences from) the higher dimensional cases which we later present.

We use the form of Kerr–AdS$_3$ metric in ref. [17] which resembles the higher dimensional Kerr–AdS metrics and can be obtained from the BTZ black hole by coordinate transformation [18,19]:

$$\begin{aligned} ds^2 = &-\frac{\Delta_r}{r^2}\left(dt - \frac{a}{\Xi}d\phi\right)^2 + \frac{r^2}{\Delta_r}dr^2 \\ &+\frac{1}{r^2}\left(adt - \frac{(r^2+a^2)}{\Xi}d\phi\right)^2 \ , \end{aligned} \tag{7}$$

where

$$\Delta_r = (r^2+a^2)\left(1+\frac{r^2}{l^2}\right) - 2MGr^2, \tag{8}$$

$$\Xi = \left(1-\frac{a^2}{l^2}\right) \ . \tag{9}$$

Here, a is the rotational parameter. The horizons are the zeros of Δ_r, and r_+ is the outer horizon which is [19]:

$$\begin{aligned} r_+^2 = &\frac{l^2}{2}\left(2MG - 1 - \frac{a^2}{l^2}\right) \\ &+\frac{l^2}{2}\sqrt{\left(1+\frac{a^2}{l^2}-2MG\right)^2 - 4\frac{a^2}{l^2}} \ . \end{aligned} \tag{10}$$

If we were to take $t\to i\tau$, defining the Euclidean section (putting the theory at finite temperature), regularity (thermal equilibrium) requires that the Euclidean time period (inverse temperature) β has the form:

$$\beta = \frac{2\pi(r_+^2+a^2)}{r_+(2r_+^2/l^2+1+a^2/l^2-2MG)} \ . \tag{11}$$

The angular velocity of the horizon is

$$\Omega = \frac{a\Xi}{r_+^2+a^2} \ , \tag{12}$$

while the area of the horizon is

$$\mathcal{A} = \frac{2\pi(r_+^2+a^2)}{r_+\Xi} \ . \tag{13}$$

The boundary metric of Kerr–AdS$_3$ is, as $r\to\infty$, given by

$$ds^2 = \frac{r^2}{l^2}\left[-dt^2 + \frac{2a}{\Xi}dtd\phi + \frac{l^2}{\Xi}d\phi^2\right] \ . \tag{14}$$

Removing the conformal factor, our dual field theory is defined on the spacetime with metric γ_{ab}:

$$[\gamma_{ab}] = \begin{pmatrix} -1 & a/\Xi \\ a/\Xi & l^2/\Xi \end{pmatrix} \ . \tag{15}$$

After some computation, the components of the stress tensor at large r are found to be

$$\begin{aligned} 8\pi G T_{tt} &= \frac{1}{2}\left(2MG - 2 + \Xi\right) + O\left(\frac{1}{r}\right), \\ 8\pi G T_{t\phi} &= -\frac{a(2MG+\Xi)}{2l\Xi)} + O\left(\frac{1}{r}\right), \\ 8\pi G T_{\phi\phi} &= \frac{(2MG(1+a^2/l^2)-\Xi^2)l}{2\Xi^2} + O\left(\frac{1}{r}\right) . \end{aligned} \tag{16}$$

It is interesting to note that the resulting field theory stress tensor (obtained using eqn.(6) and discussion below) can be written in the following form[3]:

[2]The theory on the boundary at radius r can be taken to be a dual field theory with an ultraviolet cutoff proportional to r. Then $r\to\infty$ defines the UV fixed point conformal field theory. This also fits with the fact that the counterterms, while regulating an infra–red (IR) divergence coming from the bulk, have the dual interpretation as regulating UV divergences in the field theory [16,6].

[3]We thank R. C. Myers for the suggestion that $\widehat{T}^{ab}$ might be written in this form.

$$\widehat{T}^{ab} = A\left(2u^a u^b + \gamma^{ab}\right) + B v^a v^b \qquad (17)$$

where u^a=$(1,0)$ is a unit time–like four vector and v^a=$(1, 1/l{-}a/l^2)$ is a null vector, *i.e.,* $v^b v_b$=0. The tensor is therefore manifestly traceless. The coefficients are:

$$A = \frac{\left(2MG-(1+a/l)^2\right)}{16\pi G l}; \quad B = \frac{a(1+a/l)^2}{8\pi G l^2}\ . \qquad (18)$$

It is straightforward to check that this tensor is covariantly conserved. Notice that when a=0, B vanishes and A=$(2MG{-}1)/(16\pi G l)$, giving a standard form for the stress–energy–momentum tensor of a fluid of massless particles with some energy density given by $\widehat{T}_{00}$. Notice that A, in front of the part of the stress tensor in standard form contains the black hole parameter M —the thermal part of the field theory— while B, the coefficient of the null vector part, refers only to rotational parameters. We will find that this form will persist to higher dimensions.

To calculate the conserved quantities for these spacetimes, we use the following definition for a conserved charge [20], associated to a symmetry generated by the Killing vector ξ^μ:

$$Q_\xi = \int_\Sigma d^{n-1}x\sqrt{\sigma}u^\mu T_{\mu\nu}\xi^\nu\ . \qquad (19)$$

where u_μ=$-N\,t_{,\mu}$, while N and σ are the lapse function and the spacelike metric which appear in the ADM–like decomposition of the boundary metric

$$ds^2 = -N^2dt^2 + \sigma_{ab}(dx^a + N^a dt)(dx^b + N^b dt)\ . \qquad (20)$$

Our convention for the Killing vectors ξ^μ is as follows: ∂_t is the Killing vector conjugate to the time t and ∂_ϕ is the Killing vector conjugate to ϕ. Using the above definition for the conserved charge, the mass and the angular momentum of the Kerr–AdS$_3$ spacetime are given by

$$\mathcal{M} = \frac{1}{8G\Xi}\left(2MG-\Xi\right)\ , \quad \mathcal{J} = \frac{1}{2}\frac{Ma}{\Xi^2}\ . \qquad (21)$$

Direct evaluation of the counterterms gives the finite action

$$I_3 = I_{\rm bulk} + I_{\rm surf} + I_{\rm ct} = -\frac{1}{4G}\frac{\pi(r_+^2+a^2)}{r_+\Xi}\ . \qquad (22)$$

The action with the other quantities satisfy the first law of thermodynamics

$$S = \beta(\mathcal{M}-\Omega\mathcal{J}) - I_{n+1} = \frac{\mathcal{A}}{4G}\ , \qquad (23)$$

for n=2, which is a non–trivial check of some of our computations. We will perform this check in the more complicated examples to come.

A. Comparison to Field Theory

Our dual field theory resides on the spacetime with metric $\gamma_{\mu\nu}$ giving line element

$$ds^2 = -dt^2 + \frac{2a}{\Xi}dtd\phi + \frac{l^2}{\Xi}d\phi^2\ . \qquad (24)$$

Notice that this can be brought into the form of a metric on a cylinder ($R{\times}S^1$)

$$ds^2 = -dT^2 + R^2\,d\Phi^2\ , \qquad (25)$$

with R=$l/\sqrt{\Xi}$, using the following coordinate transformation [17]:

$$T = \frac{t}{\sqrt{\Xi}},\ \Phi = \phi + at/l^2\ . \qquad (26)$$

Now, ϕ is identified with $\phi{\sim}\phi{+}\Delta\phi$, where $\Delta\phi$=$\beta\Omega$ is the period of ϕ.

Our traceless result above (eqn.'s (16) and (17)) therefore correctly reproduces the result for the trace anomaly, given in two dimensions by

$$\widehat{T}^a_a = -\frac{c\mathcal{R}}{24\pi}\ , \qquad (27)$$

where c is the central charge and $\mathcal{R}$ is the intrinsic curvature of spacetime, which is zero for the cylinder.

According to the correspondence, the Casimir energy of the dual field theory[4] is the contribution to the mass of spacetime $\mathcal{M}$ which is independent of the black hole parameter[5], M, and is given by

$$\mathcal{E} = -\frac{1}{8G}\ . \qquad (28)$$

Given the standard result [24] for the relationship between the central charge of the conformal field theory and gravity in AdS$_3$, c=$3l/2G$, we see that this translates into a vacuum energy $-c/(12l)$ for the theory on the cylinder. This is consistent with the interpretation [25] of the spacetime (7), with M=0, as the Neveu-Schwarz–Neveu-Schwarz vacuum of the holographically dual super conformal field theory: The fermions have anti–periodic boundary conditions as they go once around the cylinder, preventing their zero–point energy from cancelling that of the bosons, as happens in the Ramond–Ramond sector.

[4] In higher dimensions, connections between energy of a spacetime solution and Casimir energy of a dual field theory were first pointed out in ref. [23].

[5] In the literature, this contribution is often called the mass of the zero–mass BTZ black hole.

III. KERR–ADS$_4$

The Kerr–AdS$_4$ metric has the following form [21]

$$ds^2 = -\frac{\Delta_r}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi}d\phi\right)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{l^2}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(adt - \frac{(r^2+a^2)}{\Xi}d\phi\right)^2 , \quad (29)$$

where

$$\begin{aligned} \Delta_r &= (r^2+a^2)(1+r^2/l^2) - 2MrG , \\ \Delta_\theta &= 1-(a^2/l^2)\cos^2\theta , \\ \rho^2 &= r^2 + a^2\cos^2\theta . \end{aligned} \quad (30)$$

The period is given by

$$\beta = \frac{4\pi(r_+^2+a^2)}{r_+(3r_+^2/l^2+1+a^2/l^2-a^2/r_+^2)} , \quad (31)$$

with the angular velocity given by eqn. (12). Here, r_+ is the location of the horizon, the largest root of Δ_r.

The area of the horizon is

$$\mathcal{A} = 4\pi\left(\frac{r_+^2+a^2}{\Xi}\right) . \quad (32)$$

The non–vanishing components of the Kerr–AdS$_4$ stress tensor for large r exactly match the components recently computed for the Kerr–Newman–AdS$_4$ case in ref. [13] in the limit where the charges vanish (*i.e.,* their $z\to 0$) and are

$$8\pi G T_{tt} = \frac{2M}{rl} + O\left(\frac{1}{r^2}\right) \quad (33)$$

$$8\pi G T_{t\phi} = -\frac{2aM}{r\Xi l}\sin^2\theta + O\left(\frac{1}{r^2}\right) \quad (34)$$

$$8\pi G T_{\theta\theta} = \frac{Ml}{r\Delta_\theta} + O\left(\frac{1}{r^2}\right) \quad (35)$$

$$8\pi G T_{\phi\phi} = \frac{Ml\sin^2\theta}{r\Xi^2}[3a^2\sin^2\theta/l^2+\Xi] + O\left(\frac{1}{r^2}\right) . \quad (36)$$

The mass and the angular momentum are computed from this as:

$$\mathcal{M} = \frac{M}{\Xi};\;\; \mathcal{J} = \frac{aM}{\Xi^2} . \quad (37)$$

The boundary metric of Kerr–AdS$_4$ is given by

$$ds^2 = \frac{r^2}{l^2}\left[-dt^2 + \frac{2a\sin^2\theta}{\Xi}dtd\phi + l^2\frac{d^2\theta}{\Delta_\theta} + l^2\frac{\sin^2\theta}{\Xi}d\phi^2\right] . \quad (38)$$

Removing the conformal factor (see (5)), our dual field theory is defined on the spacetime (with coordinates (t,ϕ,θ)) with metric γ_{ab}:

$$[\gamma_{ab}] = \begin{pmatrix} -1 & a\sin^2\theta/\Xi & 0 \\ a\sin^2\theta/\Xi & l^2\sin^2\theta/\Xi & 0 \\ 0 & 0 & l^2/\Delta_\theta \end{pmatrix} . \quad (39)$$

Converting with the conformal factor and taking the limit (see (6)), we find that the stress tensor of the field theory has this simple form:

$$\widehat{T}^{ab} = \frac{M}{8\pi l^2}\left[3u^a u^b + \gamma^{ab}\right] \quad (40)$$

where $u^a = (1,0,0)$. This is the standard form for the stress tensor of the non–rotating theory 2+1 dimensional conformal field theory! The tensor is covariantly conserved and manifestly traceless, the latter result being consistent with the absence of a conformal anomaly in odd dimensional spacetime. It is interesting to note that the tensor can be written in such a simple form for this theory, even in the presence of rotation, in contrast to the case in two dimensions (see eqn.(17)), and as we will see, the four dimensional case.

The action calculation in this case gives the result:

$$I_4 = -\frac{\pi(r_+^2+a^2)(r_+^2/l^2-1)}{G(3r_+^4/l^2+r_+^2+a^2r_+^2/l^2-a^2)\Xi} , \quad (41)$$

which agrees with action calculation using the background subtraction technique in ref. [17], since the Casimir energy is zero for an odd dimensional field theory. Also it agrees with the action for Kerr–Newmann [13] in the limit where the charges vanish. These quantities also satisfy the first law (*i.e.,* eqn.(23) with n=3.)

IV. KERR–ADS$_5$

The metric for the Kerr–AdS$_5$ in general has two rotation parameters since the rotation group is $SO(4)\cong SU(2)_L\times SU(2)_R$. Here we discuss the one–parameter solution given by [17]

$$ds^2 = -\frac{\Delta_r}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi}d\phi\right)^2 + r^2\cos^2\theta d\psi^2 + \frac{l^2}{\Delta_\theta}d\theta^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\Delta_\theta\sin^2\theta}{\rho^2}\left(adt - \frac{(r^2+a^2)}{\Xi}d\phi\right)^2 . \quad (42)$$

Now we have

$$\Delta_r = (r^2+a^2)(1+r^2/l^2) - 2MG , \quad (43)$$

and the remaining quantities are as in eqn.(30). This time the inverse temperature is

$$\beta = \frac{2\pi(r_+^2+a^2)}{r_+(2r_+^2/l^2+1+a^2/l^2)} , \quad (44)$$

with the angular velocity given again by eqn. (12). Again, r_+ is the location of the horizon, the largest root of Δ_r.

In ref. [17] the calculation for the relevant physical quantities was carried out using the subtraction technique. The reference spacetime in those calculations was the spacetime with M=0, *i.e.,* AdS$_5$ in very non–standard coordinates. Here, we go some steps further, by computing and studying physics intrinsic to the Kerr–AdS$_5$ spacetime with no reference to a background, allowing us to extract physical quantities like the Casimir energy and other interesting features of the stress tensor, as we shall see.

After computation, we get the following non–vanishing components for the stress tensor at large r,

$$\begin{aligned}
8\pi G T_{tt} &= \frac{l}{8r^2}[24MG/l^2 - 14a^2\cos^2\theta/l^2 - 14a^4\cos^2\theta/l^4 \\
&\quad +15a^4\cos^4\theta/l^4 + 3(1+a^2/l^2)^2] + O\left(\frac{1}{r^4}\right) , \\
8\pi G T_{t\phi} &= \frac{al\sin^2\theta}{8r^2\Xi}[2a^2\cos^2\theta/l^2 - 7a^4\cos^4\theta/l^4 + \Xi^2 \\
&\quad +10a^4\cos^2\theta/l^4 - 24MG/l^2 - 4a^4/l^4] \\
&\quad +O\left(\frac{1}{r^4}\right) , \\
8\pi G T_{\phi\phi} &= \frac{l^3\sin^2\theta}{\Xi^2}[7a^6\cos^4/l^6 - 7a^4\cos^4\theta/l^4 + 3a^6/l^6 \\
&\quad -8a^4\cos^2\theta/l^4 - 32a^2MG\cos^2\theta/l^4 + 1 \\
&\quad -10a^6\cos^2\theta l^6 - 3a^2/l^2 + 2a^2\cos^2\theta/l^2 \\
&\quad +24a^2MG/l^4 - a^4/l^4 + 8MG/l^2] + O\left(\frac{1}{r^4}\right), \\
8\pi G T_{\theta\theta} &= \frac{l^3}{8r^2\Delta_\theta}[8MG/l^2 + \Xi^2 - 3a^4\cos^4\theta/l^4 \\
&\quad +2a^2\cos^2\theta/l^2 + 2a^4\cos^2\theta/l^4] + O\left(\frac{1}{r^4}\right) , \\
8\pi G T_{\psi\psi} &= \frac{l^3\cos^2\theta}{8r^2}[2a^2\cos^2\theta/l^2 + \Xi^2 + 8MG/l^2 \\
&\quad -7a^4\cos^4\theta/l^4 + 2a^4\cos^2\theta/l^4] + O\left(\frac{1}{r^4}\right) . \quad (45)
\end{aligned}$$

Using the above definition (19) for a conserved charge, one can calculate the mass and angular momentum of the solution, with the result:

$$\mathcal{M} = \frac{\pi l^2}{96\Xi}[a^4/l^4 + 9\Xi + 72M/l^2] \ , \quad \mathcal{J} = \frac{\pi M a}{2\Xi^2} \ . \quad (46)$$

For the action one gets

$$\begin{aligned}
I_5 &= \frac{\pi^2 l^2 (r_+^2 + a^2)}{48Gr_+(2r_+^2/l^2 + 1 + a^2/l^2)\Xi}[a^4/l^4 - 24r_+^2a^2/l^4 \\
&\quad -9a^2/l^2 - 24r_+^4/l^4 + 24M/l^2 + 9] \ . \quad (47)
\end{aligned}$$

The area of the horizon is

$$\mathcal{A} = 2\pi^2\frac{r_+(r_+^2+a^2)}{\Xi} \ . \quad (48)$$

The above quantities satisfy the first law (*i.e.,* eqn.(23) with n=4).

A. Comparison to Field Theory

The metric on the boundary is that of a rotating Einstein universe:

$$\begin{aligned}
ds^2 = \frac{r^2}{l^2}\Bigg[&-dt^2 + \frac{2a\sin^2\theta}{\Xi}dtd\phi + l^2\frac{d^2\theta}{\Delta_\theta} \\
&+l^2\frac{\sin^2\theta}{\Xi}d\phi^2 + l^2\cos^2\theta d\psi^2\Bigg] \ . \quad (49)
\end{aligned}$$

Removing the factor r^2/l^2 (defining the metric γ_{ab}) gives the line element of the space–time (with coordinates (t,ϕ,θ,ψ)) upon which our conformal field theory resides, and for which we must compute the energy–momentum tensor in order to compare to the gravity computation. This seems at first a daunting prospect, until one notices by direct computation that the Weyl tensor vanishes for this spacetime, showing that it is conformally flat. This indeed follows from the fact [17] that the spacetime (42), with M= 0, is actually just AdS$_5$ in non–standard coordinates, and so its boundary shares some of the conformal properties of the boudary of AdS$_5$.

One can therefore use the following general expression from ref. [26] to calculate the stress tensor of a field theory defined on conformally flat spacetime in four dimensions:

$$<\widehat{T}^s_{ab}> = -\frac{1}{16\pi^2}\left[\frac{1}{9}\alpha^s H_{ab}{}^{(1)} + 2\beta^s H_{ab}{}^{(3)}\right] \ , \quad (50)$$

where $H_a^{b\,(1)}$, $H_a^{b\,(3)}$, α^s and β^s are defined in ref. [26] (There, they use spacetime indices (μ,ν) for the field theory while here we use (a,b)). The label $s\in\{0,1/2,1\}$ distinguishes the spin of the field for which the labelled coefficients α^s and β^s are computed. We can now compare this to the the non–thermal (*i.e.,* the M–independent) part of the tensor which we have computed on the gravity side.

According to the AdS/CFT holographic relation, one should define the Casimir energy for the field theory dual to the Kerr–AdS spacetime as the contribution to the total energy of the spacetime (46) which is independent of the black hole's mass. This is given by

$$\mathcal{E} = \frac{\pi l^2(a^4/l^4 + 9\Xi)}{96\,G\Xi} \ . \quad (51)$$

In the limit $a \to 0$ this reduces to the Casimir energy of the non–rotating black hole discussed in ref. [6].

Now we would like to show that the expression in eqn. (51) exactly matches the energy of the $\mathcal{N}$=4 supersymmetric $U(N)$ Yang–Mills theory defined on a rotating Einstein universe, which is the CFT on the boundary. The relation between the parameters of the gravity theory in the bulk and those of the CFT on the boundary is [1]

$$\frac{1}{G} = \frac{2N^2}{\pi l^3} \ . \quad (52)$$

Using this relation in eqn. (51) one gets

$$\mathcal{E} = \frac{N^2}{48l\Xi}(a^4/l^4 + 9\Xi) \ . \tag{53}$$

Since the the gravitational quasilocal stress tensor and the stress tensor of the CFT on the boundary are dual to one another, one can use the following expression for the energy of the field theory,

$$\mathcal{E} = \sum_{s=0,\frac{1}{2},1} n^s \int_\Sigma d^3x\sqrt{\sigma}\xi^a <\widehat{T}^s_{ab}> u^b. \tag{54}$$

(Here ξ^a and u^a have the same meaning as before.) This gives

$$\mathcal{E} = \frac{4\pi^2}{\Xi}\sum_{s=0,\frac{1}{2},1} n^s\left[\frac{\alpha^s}{9}(3a^4/l^4 - 9\Xi) + \frac{\beta^s}{3}(a^4/l^4 + 9\Xi)\right] \ , \tag{55}$$

where n^s is the number of particles with spin s in the $\mathcal{N}$=4 super Yang–Mills theory on the boundary. The spin half particle is the Weyl fermion. Substituting in the values for (n^s, α^s, β^s), we find that the resulting energy of the CFT exactly matches the prediction (53) for the Casimir energy from gravity.

B. Conformal Anomaly

If we expand the the trace of quasilocal stress tensor on the gravity side in powers of $1/r$, the leading contribution is

$$T^a_a = -\frac{a^2 l}{8\pi G r^4}[a^2/l^2(3\cos^4\theta - 2\cos^2\theta) - \cos 2\theta] + O\left(\frac{1}{r^6}\right) \ . \tag{56}$$

Using the relation eqn. (52) between the gravitational parameters and the gauge theory parameters and taking the large r limit, one gets a prediction for the field theory quantity:

$$\widehat{T}^a_a = -\frac{N^2a^2}{4\pi^2l^6}[a^2/l^2(3\cos^4\theta - 2\cos^2\theta) - \cos 2\theta] \ . \tag{57}$$

This precisely matches what one obtains by performing directly the trace of the stress–tensor defined for the field theory in eqn.(50).

As a final check, we note that the general form of the conformal anomaly in four dimensions (adapted to the present case [22,6]) is given by

$$\widehat{T}^a_a = -\frac{N^2}{4\pi^2}\left[-\frac{1}{8}\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu} + \frac{1}{24}\mathcal{R}^2\right] \ , \tag{58}$$

where $\mathcal{R}_{ab}$ and $\mathcal{R}$ are the Ricci tensor and scalar for the spacetime upon which the field theory resides. Using the boundary metric γ_{ab}, we find that this is precisely the result obtained in eqn.(57).

We find that the stress–tensor for the CFT on the rotating static Einstein universe may be written in the following form:

$$\begin{aligned}\widehat{T}^{ab} &= A\,(4u^au^b + \gamma^{ab}) + B\,v^a_+v^b_+ + C\,v^a_-v^b_- \\ &\quad + D^+w^a_+w^b_+ + D^-w^a_-w^b_- + E\,z^az^b + \frac{1}{4}\gamma^{ab}\widehat{T}^d_d \ . \end{aligned}\tag{59}$$

The unit time–like velocity u^a and the accompanying null vectors are:

$$\begin{aligned} u^a &= (1,0,0,0) \ , \\ v^a_\pm &= \left(1,0,0,\pm\frac{1}{l\cos\theta}\right) \ , \qquad z^a = \left(1,-\frac{2a}{l^2},0,\frac{1}{l\cos\theta}\right) \ , \\ w^a_\pm &= \left(1,-\frac{1}{l}\left(\pm1+\frac{a}{l}\right),0,\frac{1}{l}\right) \ , \end{aligned}\tag{60}$$

while the coefficients are:

$$\begin{aligned} A &= \frac{1}{64\pi Gl}[a^4/l^4(3\cos^4\theta - \cos 2\theta) + 8MG/l^2 \\ &\qquad + 2\Delta_\theta - 1] \ , \\ B &= \frac{\cos\theta}{32\pi Gl(1-\cos\theta)}[a^4/l^4(3\cos^4\theta - \cos^3\theta - \cos 2\theta) \\ &\qquad + 2\Delta_\theta - 1] \ , \\ C &= -\frac{a^2\cos\theta}{32\pi Gl^3}[a^2/l^2(3\cos^3\theta - \cos\theta - \cos 2\theta) \\ &\qquad + 1 - \cos\theta] \ , \\ D_\pm &= -\frac{a(1\pm a/l)}{32\pi Gl^2(1-\cos\theta)}[a^3/l^3(\cos 2\theta\cos\theta + \sin^2\theta) \\ &\qquad + a\cos\theta/l \pm \Delta_\theta] \ , \\ E &= -\frac{\cos\theta}{32\pi Gl(1-\cos\theta)}\Xi\Delta_\theta \ . \end{aligned}\tag{61}$$

Notice again, just as we saw in lower dimensions, that the coefficient, A, of the part of the tensor in standard form is the only one which depends upon the thermal part of the theory —represented by the black hole mass— while the other coefficients, referring to the null vectors, are independent of M. It is also interesting to note that when $a\to0$, while C and $D^\pm$ vanish, the terms involving B and E cancel each other.

The last term in the energy–momentum tensor (59) emphasizes the conformal anomaly's presence. That there is a conformal anomaly is interesting. General considerations (see *e.g.,* ref. [22]) show that in the presence of a conformal anomaly, the action has a divergent piece

$$I_{\rm div} \propto \log(r/l)\int d^4x\sqrt{-\gamma}\,\widehat{T}^a_a \ . \tag{62}$$

(See ref. [7] for specific examples in this context.) From reading ref. [28], however, one expects that the corresponding logarithmic (UV) divergence ($\log(r/l)$) of the action should not be present for a spacetime which can be written locally as a product. In support of this, the action (47) which we computed does not have a logarithm (in contrast to the examples in ref. [7]), as we have seen, presenting us with a paradox. This is resolved upon realization that the *integrated* trace actually does vanish, as a computation reveals[6]. It would be interesting to characterize this example further, and find others of this type.

We see that the gravity computation (which is dual to a computation in *strongly coupled* field theory) matches the results from the weakly coupled field theory computation. In the case of pure AdS_5, the reason for the precise match, as explained in ref. [6], is that pure AdS_5 receives no stringy corrections because all tensors (in terms of which the stringy corrections can be written) that can modify Einstein's field equations vanish for that space. This translates into no difference between the results for strong and weak coupling field theory, because the stringy corrections are correlated with corrections in field theory coupling, as required by the duality [1].

That this is also true for the Kerr–AdS_5 case, in the limit of zero black hole mass —where we have shown full agreement with field theory— follows simply because M=0 Kerr–AdS_5 is just AdS_5 in non–standard coordinates [17], parameterised by a. It therefore shares many of the conformal properties with AdS_5. As the stringy corrections can all be written in terms of the Weyl tensor of the spacetime, if they vanish for AdS_5, they vanish here also.

V. CONCLUDING REMARKS

The counterterm subtraction technique has allowed for the intrinsic definition of the action (and the quantities which follow) for the Kerr–AdS_{n+1} spacetimes, for n=2, 3, 4. This is in contrast to the background subtraction technique, which required a reference spacetime in order to compute a finite action. In subtracting the contribution from a reference spacetime to get a finite action, any physics common to both spacetimes is lost. In this paper we have computed examples of such quantities, the Casimir energies and conformal anomaly for field theories residing on even dimensional spacetimes.

We have shown that the quantities thus computed (and indeed, the full stress–energy–momentum tensor in four dimensions) are consistent with a holographic interpretation of the physics of gravity in AdS in terms of dual conformal field theory. The Casimir energies which we computed for the holographically dual two and four dimensional conformal field theories (residing on a rotating cylinder, or the rotating Einstein universe) exactly matched the contribution to the energy computed for the Kerr–AdS spacetime which is not attributable to the black hole. Also, the non–zero conformal anomaly in the four dimensional case exactly matched the result for Kerr–AdS_5. In all cases (n=2, 3, 4) we found that the stress tensor for the field theory on the rotating spacetime can be neatly written in terms of a sum of a standard traceless thermal form (involving the mass of the dual black hole) and additional terms involving null vectors. Intriguingly, for n=3, there are no such additional terms; its stress tensor keeps the simple thermal form. This may be an important feature of the M2–brane world–volume conformal field theory [29,30]. While the computation for the two dimensional conformal field theory and the related AdS_3 spacetime is essentially contained in ref. [6], the results for higher dimensions are new.

We expect that these results also follow for Kerr–AdS_{n+1}, for $n+1>5$. It would be especially interesting to see the form of the tensor for Kerr–AdS_7, including its trace. However, we found that the computational complexities encountered due to the off–diagonal terms in the metric —and the additional counterterms needed [7]— were too severe to allow for an exploration using the techniques which we have employed so far.

ACKNOWLEDGEMENTS

We would like to thank Roberto Emparan, Robert C. Myers and Harvey Reall for discussions, and especially RE and RCM for suggestions and comments on a preliminary draft of this paper. CVJ would like to thank the Relativity group at D.A.M.T.P., Cambridge (UK) for hospitality during the early stages of this work. This work was supported by an NSF Career grant, #PHY–9733173. This paper is report #'s UK/99–13, IASSNS–HEP–99/88, and DTP/99/71.

[6] We are grateful to R. C. Myers for a crucial discussion, and for pointing out the vanishing.

♯ CVJ is on leave from the U. of K.

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